1. Calculate the following real integrals (you might want to use the residue theorem for that nevertheless):

(a) $\displaystyle\int_0^{2\pi} \frac{\cos(\theta)\sin(2\theta)}{5+3\cos(2\theta)}\,d\theta.$

(b) $\displaystyle\int_0^{2\pi} \frac{\sin^2(\frac{5\theta}{2})}{\sin^2(\frac{\theta}{2})}\,d\theta.$

Hint: In order to avoid obtaining expressions of the form $z^{\frac{1}{2}}$ *in the complex integral, you might want to use some trigonometric identities to substitute* $\sin^2(w)$ *with a formula involving* $\cos(2w)$.

(c) For $0 < p < 1$: $\displaystyle\int_0^{2\pi} \frac{1}{1-2p\cos(\theta)+p^2}\,d\theta.$

2. Use the residue theorem to compute the Fourier transform $\hat{f}(a)$ of the function

$$f(x) = \frac{x}{1+x^4}.$$

3. In this exercise, we will show that the Fourier transform of a Gaussian function is again a Gaussian function. Let

$$f(x) = e^{-\frac{x^2}{2}}.$$

(a) Using an appropriate change of variables, show that

$$\hat{f}(a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-iax}e^{-\frac{x^2}{2}}\,dx = \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}}\lim_{L\to+\infty}\int_{-L+ia}^{L+ia} e^{-\frac{z^2}{2}}\,dz,$$

where the final integral is considered over the line $\mathrm{Im}(z) = a$ in the complex plane.

(b) Show, using the techniques that we learned about moving the curve of integration in the complex plane for integrals of holomorphic functions, that

$$\lim_{L\to+\infty}\int_{-L+ia}^{L+ia} e^{-\frac{z^2}{2}}\,dz = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\,dx.$$

Hint: In the case $a \geqslant 0$ (the case $a < 0$ is completely analogous), you might want to apply Cauchy's theorem to the rectangle which is the boundary of $\{-L \leqslant \mathrm{Re}(z) \leqslant L\} \cap \{0 \leqslant \mathrm{Im}(z) \leqslant a\}$. What happens to the integral over the edges at $\mathrm{Re}(z) = \pm L$ as $L \to +\infty$?

(c) Use polar coordinates to compute the integral

$$\Big(\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\,dx\Big)^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{x^2+y^2}{2}}\,dxdy.$$

(d) Combining the above calculations, show that

$$\hat{f}(a) = f(a).$$

(e) Using the general properties of the Fourier transform, show that, for any $\sigma \neq 0$,

$$\mathcal{F}[e^{-\frac{x^2}{2\sigma^2}}](a) = |\sigma| e^{-\frac{\sigma^2 a^2}{2}}.$$

(4.) Let $f : [0, +\infty) \to \mathbb{C}$ be a piecewise continuous function.

(a) Suppose that there exists some $\gamma_0 \in \mathbb{R}$ such that we have

$$\int_0^{+\infty} |f(t)| e^{-\gamma_0 t}\,dt < +\infty.$$

Show that, for any $\gamma > \gamma_0$, we have for the function $t \cdot f(t)$:

$$\int_0^{+\infty} |t \cdot f(t)| e^{-\gamma t}\,dt < +\infty.$$

(Hint: Show that $t \leqslant Ce^{(\gamma-\gamma_0)t}$ for some constant $C > 0$ independent of t.)

(b) Recall that a $\gamma_0 \in \mathbb{R}$ is called an *abscissa of convergence* for the Laplace transform $\mathcal{L}[f](z)$ of f if, for every $z \in \mathbb{C}$ with $\mathrm{Re}(z) > \gamma_0$, the integral defining $\mathcal{L}[f](z)$ is well-defined ("converges absolutely"), that is to say:

$$\int_0^{+\infty} |f(t)||e^{-zt}|\,dt < +\infty.$$

Using the previous part of the exercise, show that, if γ_0 is an abscissa of convergence for $\mathcal{L}[f(t)]$, then γ_0 is also an abscissa of convergence for $\mathcal{L}[t \cdot f(t)]$ (and, therefore, by repeating the same process, also for $\mathcal{L}[t^n f(t)]$ for any $n \in \mathbb{N}$).

Solutions

1. **(a)** Let $z = e^{i\theta}$, $\theta \in [0, 2\pi]$, be the parametrization of the unit circle $\gamma = \{|z| = 1\}$, so that:

$$\cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin(2\theta) = \frac{1}{2i}\left(z^2 - \frac{1}{z^2}\right), \quad \cos(2\theta) = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right), \quad d\theta = \frac{dz}{iz}$$

Then, the numerator becomes:

$$\cos(\theta)\sin(2\theta) = \left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right)\left(\frac{1}{2i}\left(z^2 - \frac{1}{z^2}\right)\right) = \frac{1}{4i}(z^3 + z - z^{-1} - z^{-3})$$

And the denominator:

$$5 + 3\cos(2\theta) = \frac{10 + 3z^2 + 3z^{-2}}{2}$$

So the integrand becomes:

$$\left(\frac{1}{4i} \cdot \frac{z^3 + z - z^{-1} - z^{-3}}{\frac{1}{2}(10 + 3z^2 + 3z^{-2})}\right) \cdot \frac{dz}{iz} = -\frac{1}{2} \cdot \frac{z^3 + z - z^{-1} - z^{-3}}{z(10 + 3z^2 + 3z^{-2})}\, dz$$

Now we evaluate:

$$\int_\gamma \frac{z^3 + z - z^{-1} - z^{-3}}{z(10 + 3z^2 + 3z^{-2})}\, dz \doteq \int_\gamma g(z)\frac{dz}{z}, \quad g(z) = \frac{z^3 + z - z^{-1} - z^{-3}}{z(10 + 3z^2 + 3z^{-2})}.$$

We could proceed by finding the roots of the denominator inside the unit disc, evaluate the residues etc, but there is a faster way: Note that the inversion $z \to w = \frac{1}{z}$ maps the unit circle to the unit circle with the *opposite parametrisation* (since if $z = e^{i\theta}$, then $w = e^{-i\theta}$), and we have

$$g(w) = g\left(\frac{1}{z}\right) = -g(z)$$

and

$$\frac{dz}{z} = -\frac{dw}{w}.$$

Therefore,

$$I = \int_\gamma g(z)\frac{dz}{z} = \int_{-\gamma} (-g(w))\left(-\frac{dw}{w}\right) = -I,$$

so $I = 0$.

(b) We begin by using the identity:

$$\sin^2 x = \frac{1 - \cos(2x)}{2},$$

to write:

$$\frac{\sin^2\left(\frac{5\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)} = \frac{1 - \cos(5\theta)}{1 - \cos(\theta)}.$$

Now we change variables using $z = e^{i\theta}$, so that:

$$\cos(n\theta) = \frac{1}{2}\left(z^n + z^{-n}\right), \quad d\theta = \frac{dz}{iz}.$$

Thus, we have:

$$1 - \cos(5\theta) = 1 - \frac{1}{2}(z^5 + z^{-5}) = \frac{2 - z^5 - z^{-5}}{2},$$

$$1 - \cos(\theta) = \frac{2 - z - z^{-1}}{2}.$$

So the integrand becomes:

$$\frac{1 - \cos(5\theta)}{1 - \cos(\theta)} = \frac{2 - z^5 - z^{-5}}{2 - z - z^{-1}},$$

and the integral becomes:

$$\int_0^{2\pi} \frac{2 - z^5 - z^{-5}}{2 - z - z^{-1}} \cdot \frac{dz}{iz}.$$

Multiply numerator and denominator by z^5 to remove negative powers:

$$= \frac{1}{i} \oint_{|z|=1} \frac{2z^5 - z^{10} - 1}{z(2z^5 - z^6 - z^4)}\, dz.$$

Factor the denominator:

$$2z^5 - z^6 - z^4 = z^4(2z - z^2 - 1) = -z^4(z-1)^2.$$

So the integrand becomes:

$$f(z) = \frac{2z^5 - z^{10} - 1}{-iz^5(z-1)^2}.$$

At first glance, it might seem that we are integrating through the singularity at $z = 1$. However, this is a regular point, since the numerator factorizes as $2z^5 - z^{10} - 1 = -(z^5 - 1)^2$ so we have

$$f(z) = \frac{2z^5 - z^{10} - 1}{-iz^5(z-1)^2} = \frac{(z^5-1)^2}{iz^5(z-1)^2} = \frac{\left((z-1)(z^4 + z^3 + z^2 + z + 1)\right)^2}{iz^5(z-1)^2} = \frac{(z^4 + z^3 + z^2 + z + 1)^2}{iz^5}.$$

We now compute the integral using the residue theorem:

$$\int_{\gamma = \{|z|=1\}} f(z)\, dz = 2\pi i \operatorname{Res}_{z=0} f.$$

Since f is expressed already in powers of z, we can immediately calculate the residue by finding the coeffcient of the z^{-1} term:

$$f(z) = \frac{(z^4 + z^3 + z^2 + z + 1)^2}{iz^5}$$

$$= \frac{z^8 + z^6 + z^4 + z^2 + 1 + 2z^7 + 2z^6 + 2z^5 + 2z^4 + 2z^5 + 2z^4 + 2z^3 + 2z^3 + 2z^2 + 2z}{iz^5}$$
$$= -i\big(z^{-5} + 2z^{-4} + 3z^{-3} + 4z^{-2} + 5z^{-1} + 4 + 3z + 2z^2 + z^3\big),$$

so $\mathrm{Res}_{z=0} f = -5i$ and, thus,

$$\int_\gamma f(z)\,dz = 10\pi.$$

(c) We use the substitution $z = e^{i\theta}$, which gives:

$$\cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad d\theta = \frac{dz}{iz}.$$

The integrand becomes:

$$\frac{1}{1 - 2p\cos(\theta) + p^2} = \frac{1}{1 - p(z + z^{-1}) + p^2}.$$

Multiply numerator and denominator by z to eliminate negative powers:

$$\frac{1}{1 - p(z + z^{-1}) + p^2} \cdot \frac{dz}{iz} = \frac{1}{i} \cdot \frac{dz}{z(1 - pz - \frac{p}{z} + p^2)}.$$

$$= \frac{1}{i} \cdot \frac{dz}{-pz^2 + (1 + p^2)z - p}.$$

Let

$$f(z) = \frac{1}{-pz^2 + (1 + p^2)z - p}.$$

We now compute the contour integral:

$$\oint_{|z|=1} \frac{f(z)}{i}\,dz = \frac{2\pi i}{i} \cdot \mathrm{Res}_{z=z_0} f(z) = 2\pi \cdot \mathrm{Res}_{z=z_0} f(z),$$

where z_0 is the pole *inside* the unit circle.
To find the poles, solve:

$$-pz^2 + (1 + p^2)z - p = 0.$$

Using the quadratic formula:

$$z = \frac{1 + p^2 \pm \sqrt{(1 + p^2)^2 - 4p^2}}{2p} = \frac{1 + p^2 \pm (1 - p^2)}{2p}.$$

This gives two roots:

$$z_1 = \frac{1 + p^2 + (1 - p^2)}{2p} = \frac{2}{2p} = \frac{1}{p}, \quad \text{and} \quad z_2 = \frac{1 + p^2 - (1 - p^2)}{2p} = \frac{2p^2}{2p} = p.$$

Since $0 < p < 1$, we have:

- $z = p$ lies **inside** the unit circle,
- $z = \frac{1}{p}$ lies **outside** the unit circle.

Thus, the only pole inside the unit circle is at $z = p$, and it is a **simple pole**. To compute its residue, we use:

$$\text{Res}_{z=p} f(z) = \lim_{z \to p} (z-p) f(z).$$

Factor the denominator:

$$-pz^2 + (1+p^2)z - p = -p(z-p)\left(z - \frac{1}{p}\right),$$

so:

$$f(z) = \frac{1}{-p(z-p)\left(z - \frac{1}{p}\right)}.$$

Then:

$$(z-p)f(z) = \frac{1}{-p\left(z - \frac{1}{p}\right)}, \quad \text{so} \quad \lim_{z \to p}(z-p)f(z) = \frac{1}{-p\left(p - \frac{1}{p}\right)}.$$

Simplify:

$$= \frac{1}{-p\left(\frac{p^2-1}{p}\right)} = \frac{1}{-(p^2-1)} = \frac{1}{1-p^2}.$$

Thus, the integral is:

$$\int_0^{2\pi} \frac{1}{1 - 2p\cos\theta + p^2}\, d\theta = 2\pi \cdot \frac{1}{1-p^2} = \boxed{\frac{2\pi}{1-p^2}}.$$

2. We want to compute the Fourier transform of the function

$$f(x) = \frac{x}{1+x^4},$$

that is,

$$\hat{f}(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x}{1+x^4} e^{-iax}\, dx, \quad a \in \mathbb{R}.$$

We consider the complex function

$$f(z) = \frac{ze^{-iaz}}{1+z^4}.$$

The poles of $f(z)$ are the solutions to $1 + z^4 = 0 \Rightarrow z^4 = -1$, which are:

$$z_k = e^{i\frac{(2k+1)\pi}{4}}, \quad k = 0, 1, 2, 3.$$

These are explicitly:

$$z_1 = e^{i\pi/4}, \quad z_2 = e^{i3\pi/4}, \quad z_3 = e^{i5\pi/4}, \quad z_4 = e^{i7\pi/4}.$$

Upper half-plane poles: z_1, z_2
Lower half-plane poles: z_3, z_4

The poles are simple (since $z^4 + 1 = (z - z_1)(z - z_2)(z - z_3)(z - z_4)$. The residue at z_k can be computed via the formula $\operatorname{Res}_{z=z_k}(f) = \lim_{z \to z_k} ((z - z_k) f(z))$. Alternatively, we can directly use the formulas from Exercise 4 from Series 7, for functions of the form $\frac{p(z)}{q(z)}$ (which says that, in the case when $p(z_0) \neq 0$ and z_0 is a simple root of $q(z)$, we have $\operatorname{Res}_{z=z_0}(f) = \frac{p(z_0)}{q'(z_0)}$):

$$\operatorname{Res}_{z=z_k}(f) = \frac{z_k e^{-iaz_k}}{\frac{d}{dz}(1 + z^4)\Big|_{z=z_k}} = \frac{z_k e^{-iaz_k}}{4z_k^3} = \frac{e^{-iaz_k}}{4z_k^2}.$$

We close the contour in the complex plane according to the sign of a:

– If $a \leqslant 0$, then e^{-iaz} decays in the **upper half-plane** (since, for $z = x + iy$, we have $|e^{-iaz}| = |e^{-iax+ay}| = e^{ay}$, so $|e^{-iaz}| \leqslant 1$ when $a \leqslant 0$ and $y \geqslant 0$). So we close the contour in the upper half-plane, enclosing z_1, z_2, similarly to what we have already seen in class: For $R > 0$ tending to ∞, we construct the closed loop which goes from $-R$ to R on the real axis and then follows the half circle of radius R on the upper hal-plane. Following this process, as we have seen in the class, we get:

$$\int_{-\infty}^{\infty} \frac{x}{1 + x^4} e^{-iax}\, dx = 2\pi i \left(\operatorname{Res}_{z=z_1}(f) + \operatorname{Res}_{z=z_2}(f)\right) = 2\pi i \left(\frac{e^{-iaz_1}}{4z_1^2} + \frac{e^{-iaz_2}}{4z_2^2} \right).$$

– If $a > 0$, then e^{-iaz} decays in the **lower half-plane**. So we close the contour in the lower half-plane, enclosing z_3, z_4.

Important remark. When closing the loop in the lower half plane, if we give our curve a positive (i.e. counterclockwise) orientation, then the part of the curve on the real line is parametrized from $+R$ to $-R$ (make a drawing to verify this). Thus, the original integral (which corresponds to a parametrization going from $-R$ to $+R$ is *minus* the result obtained from the positively oriented loop. This explains the $-$ sign below (in comparison to what we would naively expect from the residue theorem).

Thus, we get

$$\int_{-\infty}^{\infty} \frac{x}{1 + x^4} e^{-iax}\, dx = -2\pi i \left(\operatorname{Res}_{z=z_3}(f) + \operatorname{Res}_{z=z_4}(f)\right) = -2\pi i \left(\frac{e^{-iaz_3}}{4z_3^2} + \frac{e^{-iaz_4}}{4z_4^2} \right).$$

$$\boxed{\hat{f}(a) = \begin{cases} \sqrt{2\pi} i \left(\frac{e^{-iaz_1}}{4z_1^2} + \frac{e^{-iaz_2}}{4z_2^2} \right), & a \leqslant 0, \\ -\sqrt{2\pi} i \left(\frac{e^{-iaz_3}}{4z_3^2} + \frac{e^{-iaz_4}}{4z_4^2} \right), & a > 0, \end{cases}}$$

where

$$z_k = e^{i\frac{(2k+1)\pi}{4}}, \quad k = 0, 1, 2, 3.$$

3. (a) We want to compute the Fourier transform of the Gaussian function

$$f(x) = e^{-\frac{x^2}{2}},$$

which is defined as

$$\hat{f}(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iax} e^{-\frac{x^2}{2}}\, dx, \quad a \in \mathbb{R}.$$

We perform a change of variables by setting

$$z = x + ia, \quad \text{so that} \quad x = z - ia, \quad \text{and} \quad dz = dx.$$

Thus, as x runs from $-\infty$ to $+\infty$, the variable z runs along the horizontal line $\mathrm{Im}(z) = a$ in the complex plane. Now substitute:

$$x^2 = (z - ia)^2 = z^2 - 2iaz - a^2.$$

Hence,

$$e^{-\frac{x^2}{2}} = e^{-\frac{1}{2}(z^2 - 2iaz - a^2)} = e^{-\frac{z^2}{2}} e^{iaz} e^{-\frac{a^2}{2}}.$$

Substituting into the Fourier transform expression:

$$\hat{f}(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iaz} e^{-\frac{z^2}{2}} e^{iaz} e^{-\frac{a^2}{2}}\, dz.$$

Simplifying the exponentials:

$$e^{-iaz} \times e^{iaz} = 1,$$

thus:

$$\hat{f}(a) = \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} \int_{-\infty+ia}^{+\infty+ia} e^{-\frac{z^2}{2}}\, dz.$$

Introducing the limit as $L \to +\infty$, we can write:

$$\boxed{\hat{f}(a) = \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} \lim_{L\to+\infty} \int_{-L+ia}^{L+ia} e^{-\frac{z^2}{2}}\, dz}.$$

(b) We want to show that

$$\lim_{L\to+\infty} \int_{-L+ia}^{L+ia} e^{-\frac{z^2}{2}}\, dz = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\, dx$$

by moving the contour of integration.

First of all, consider the rectangle in the complex plane with vertices at

$$-L, \quad L, \quad L + ia, \quad -L + ia.$$

Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ denote the four edges:

* γ_1: from $-L$ to L along the real axis,
* γ_2: from L to $L+ia$ vertically,
* γ_3: from $L+ia$ to $-L+ia$ horizontally,
* γ_4: from $-L+ia$ down to $-L$ vertically.

Since $e^{-\frac{z^2}{2}}$ is an entire function, by Cauchy's theorem, the integral over the boundary vanishes:

$$\int_{\partial R_L} e^{-\frac{z^2}{2}}\,dz = 0,$$

that is,

$$\int_{\gamma_1} e^{-\frac{z^2}{2}}\,dz + \int_{\gamma_2} e^{-\frac{z^2}{2}}\,dz + \int_{\gamma_3} e^{-\frac{z^2}{2}}\,dz + \int_{\gamma_4} e^{-\frac{z^2}{2}}\,dz = 0.$$

Thus:

$$\int_{\gamma_3} e^{-\frac{z^2}{2}}\,dz = -\left(\int_{\gamma_1} e^{-\frac{z^2}{2}}\,dz + \int_{\gamma_2} e^{-\frac{z^2}{2}}\,dz + \int_{\gamma_4} e^{-\frac{z^2}{2}}\,dz\right).$$

- Along γ_1, $z = x \in \mathbb{R}$, so

$$\int_{\gamma_1} e^{-\frac{z^2}{2}}\,dz = \int_{-L}^{L} e^{-\frac{x^2}{2}}\,dx.$$

As $L \to +\infty$, we have:

$$\lim_{L\to+\infty}\int_{-L}^{L} e^{-\frac{x^2}{2}}\,dx = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\,dx.$$

- Along γ_2 and γ_4 (the vertical sides), parametrize:

$$\gamma_2: \quad z = L+iy, \quad y\in[0,a],$$

$$\gamma_4: \quad z = -L+iy, \quad y\in[a,0].$$

The modulus of the integrand is (for $z = x+iy$):

$$\left|e^{-\frac{z^2}{2}}\right| = \left|e^{-\frac{1}{2}(x^2+2xyi-y^2)}\right| = \left|e^{-\frac{1}{2}(x^2-y^2)}e^{xyi}\right| = e^{-\frac{1}{2}(x^2-y^2)}$$

where $x = \pm L$.
Thus:

$$\left|e^{-\frac{z^2}{2}}\right| = e^{-\frac{1}{2}(L^2-y^2)} \leqslant e^{-\frac{1}{2}(L^2-a^2)} = e^{-\frac{L^2}{2}}e^{\frac{a^2}{2}}.$$

Therefore, the integrals over γ_2 and γ_4 satisfy:

$$\left|\int_{\gamma_2} e^{-\frac{z^2}{2}}\,dz\right| \leqslant a e^{-\frac{L^2}{2}}e^{\frac{a^2}{2}},$$

$$\left|\int_{\gamma_4} e^{-\frac{z^2}{2}}\,dz\right| \leqslant a e^{-\frac{L^2}{2}}e^{\frac{a^2}{2}}.$$

Both go to zero exponentially fast as $L \to +\infty$.

Thus, taking the limit $L \to +\infty$, the contributions of γ_2 and γ_4 vanish, and we obtain:

$$\lim_{L\to+\infty} \int_{-L+ia}^{L+ia} e^{-\frac{z^2}{2}}\, dz = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\, dx,$$

as required.

(c) We change variables to polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta,$$

so that

$$x^2 + y^2 = r^2, \quad \text{and} \quad dx\, dy = r\, dr\, d\theta.$$

Thus, the double integral becomes

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\frac{x^2+y^2}{2}}\, dx\, dy = \int_0^{2\pi} \int_0^{+\infty} e^{-\frac{r^2}{2}} r\, dr\, d\theta.$$

First, we compute the integral over r:

$$\int_0^{+\infty} r e^{-\frac{r^2}{2}}\, dr.$$

Make the substitution

$$u = \frac{r^2}{2}, \quad \text{so that} \quad du = r\, dr.$$

Thus,

$$\int_0^{+\infty} r e^{-\frac{r^2}{2}}\, dr = \int_0^{+\infty} e^{-u}\, du = \left[-e^{-u}\right]_0^{+\infty} = 1.$$

Next, we compute the integral over θ:

$$\int_0^{2\pi} d\theta = 2\pi.$$

Multiplying the two results, we find

$$\left(\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\, dx\right)^2 = 2\pi.$$

Thus, taking the square root of both sides,

$$\boxed{\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}}.$$

(d) From part (a), we have

$$\hat{f}(a) = \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} \times \lim_{L\to+\infty} \int_{-L+ia}^{L+ia} e^{-\frac{z^2}{2}}\, dz.$$

Using the results from parts (b) and (c),

$$\lim_{L\to+\infty} \int_{-L+ia}^{L+ia} e^{-\frac{z^2}{2}}\, dz = \sqrt{2\pi},$$

we substitute and simplify:

$$\hat{f}(a) = \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} \times \sqrt{2\pi} = e^{-\frac{a^2}{2}}.$$

Since

$$f(a) = e^{-\frac{a^2}{2}},$$

we conclude that

$$\boxed{\hat{f}(a) = f(a)}.$$

(e) We need to compute the Fourier transform of the function

$$g(x) = e^{-\frac{x^2}{2\sigma^2}} \quad \text{for} \quad \sigma \neq 0.$$

Notice that

$$g(x) = f\left(\frac{x}{\sigma}\right),$$

where

$$f(x) = e^{-\frac{x^2}{2}}.$$

Thus, $g(x)$ is a scaled version of $f(x)$. Recall the general scaling property of the Fourier transform:

$$\mathcal{F}[f(\lambda x)](a) = \frac{1}{|\lambda|}\hat{f}\left(\frac{a}{\lambda}\right),$$

for any $\lambda \neq 0$.
Here, $\lambda = \frac{1}{\sigma}$, so

$$\mathcal{F}[g](a) = |\sigma|\, \mathcal{F}[f](\sigma a).$$

From previous parts, we know that

$$\mathcal{F}[f](a) = e^{-\frac{a^2}{2}}.$$

Thus,

$$\mathcal{F}[f](\sigma a) = e^{-\frac{(\sigma a)^2}{2}} = e^{-\frac{\sigma^2 a^2}{2}}.$$

Substituting back, we find:

$$\mathcal{F}[g](a) = |\sigma| e^{-\frac{\sigma^2 a^2}{2}}.$$

Thus, we have shown:

$$\boxed{\mathcal{F}\left(e^{-\frac{x^2}{2\sigma^2}}\right)(a) = |\sigma| e^{-\frac{\sigma^2 a^2}{2}}}.$$

4. (a) We are given a piecewise continuous function $f : [0, +\infty) \to \mathbb{C}$ such that there exists $\gamma_0 \in \mathbb{R}$ with

$$\int_0^{+\infty} |f(t)| e^{-\gamma_0 t}\, dt < +\infty.$$

We need to show that, for any $\gamma > \gamma_0$,

$$\int_0^{+\infty} |tf(t)| e^{-\gamma t}\, dt < +\infty.$$

Since $\gamma > \gamma_0$, the quantity $\gamma - \gamma_0 > 0$.
We claim that there exists a constant $C > 0$ such that for all $t \geqslant 0$,

$$t \leqslant C e^{(\gamma - \gamma_0)t}.$$

Proof of the claim: Consider the function

$$g(t) = \frac{t}{e^{(\gamma - \gamma_0)t}}.$$

Observe that:

- $g(0) = 0$,
- As $t \to +\infty$, $g(t) \to 0$ as can be verified easily using e.g. L'Hopital's rule (in general the exponential grows faster than any polynomial),
- g is continuous on $[0, +\infty)$,
- Therefore, $g(t)$ achieves a maximum value $M \geqslant 0$ on $[0, +\infty)$.

Thus, for all $t \geqslant 0$,

$$g(t) \leqslant M,$$

that is,

$$t \leqslant M e^{(\gamma - \gamma_0)t}.$$

Setting $C = M$ completes the proof of the claim.
Using the estimate $t \leqslant C e^{(\gamma - \gamma_0)t}$, we have

$$|tf(t)| e^{-\gamma t} = |f(t)| t e^{-\gamma t} \leqslant C |f(t)| e^{-(\gamma_0)t}.$$

Thus,

$$\int_0^{+\infty} |tf(t)|e^{-\gamma t}\,dt \leqslant C \int_0^{+\infty} |f(t)|e^{-\gamma_0 t}\,dt.$$

By assumption, the right-hand side is finite:

$$\int_0^{+\infty} |f(t)|e^{-\gamma_0 t}\,dt < +\infty.$$

Therefore,

$$\boxed{\int_0^{+\infty} |tf(t)|e^{-\gamma t}\,dt < +\infty}$$

for any $\gamma > \gamma_0$.

(c) Suppose that $\gamma_0 \in \mathbb{R}$ is an abscissa of convergence for the Laplace transform $\mathcal{L}[f](z)$, that is, for every $z \in \mathbb{C}$ with $\mathrm{Re}(z) > \gamma_0$,

$$\int_0^{+\infty} |f(t)||e^{-zt}|\,dt = \int_0^{+\infty} |f(t)|e^{-\mathrm{Re}(z)t}\,dt < +\infty.$$

For any $\gamma_1 > \gamma_0$, choosing a z with $\mathrm{Re}(z) = \gamma_1$, the above implies that

$$\int_0^{+\infty} |f(t)|e^{-\gamma_1 t}\,dt < +\infty \quad \text{for any } \ \gamma_1 > \gamma_0. \tag{1}$$

We need to show that γ_0 is also an abscissa of convergence for $\mathcal{L}[t \cdot f(t)](z)$, that is,

$$\int_0^{+\infty} |tf(t)|e^{-\mathrm{Re}(z)t}\,dt < +\infty \quad \text{for all} \quad \mathrm{Re}(z) > \gamma_0.$$

For any $z \in \mathbb{C}$ with $\mathrm{Re}(z) = \gamma > \gamma_0$, let us choose some $\gamma_1 \in (\gamma_0, \gamma)$. From part (a), we know that if

$$\int_0^{+\infty} |f(t)|e^{-\gamma_1 t}\,dt < +\infty,$$

then for any $\gamma > \gamma_1$,

$$\int_0^{+\infty} |tf(t)|e^{-\gamma t}\,dt < +\infty.$$

Thus, since $\mathrm{Re}(z) = \gamma$ and the first inequality is true in view of (1), we deduce the requied estimate

$$\int_0^{+\infty} |tf(t)|e^{-\mathrm{Re}(z)t}\,dt < +\infty.$$

Thus, the Laplace transform $\mathcal{L}[tf(t)](z)$ converges absolutely for all $\mathrm{Re}(z) > \gamma_0$.

By applying the same reasoning iteratively, we conclude that if γ_0 is an abscissa of convergence for $\mathcal{L}[f](z)$, then it is also an abscissa for $\mathcal{L}[t^n f(t)](z)$ for all $n \in \mathbb{N}$